

7.3

Angle Relationships in Polygons

convex polygon

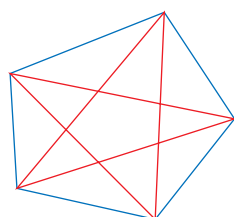
- a polygon with no part of any line segment joining two points on the polygon outside the polygon

concave polygon

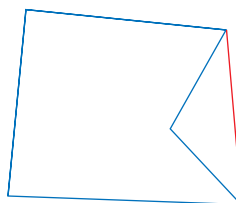
- a polygon with parts of some line segments joining two points on the polygon outside the polygon

Although triangles and quadrilaterals are the most common shapes in construction, you will find many other types of polygons in both natural and manufactured objects.

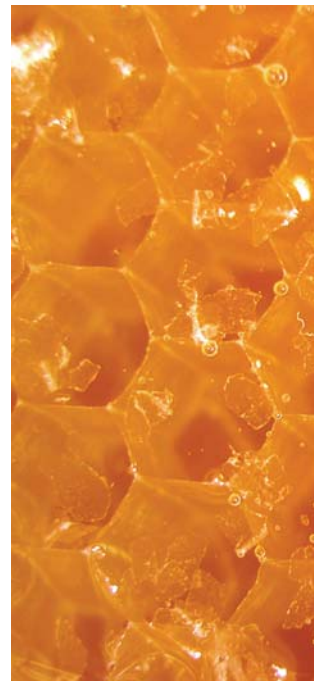
The investigations in this section deal with **convex polygons**. All the interior angles of these polygons measure less than 180° . A **concave polygon** can have angles greater than 180° .



convex polygon



concave polygon



- ruler
- protractor

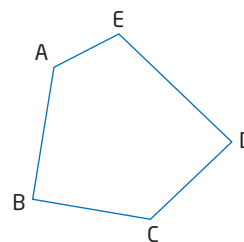
pentagon

- a polygon with five sides

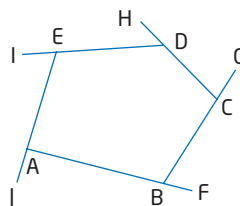
Investigate A

How are the angles in a pentagon related?

- Make a hypothesis about the sum of the interior angles of a **pentagon**.
- Draw a pentagon. Then, label and measure the five interior angles.
- Find the sum of the five interior angles. Compare this sum to the sums found by your classmates.
- Reflect** Discuss whether the sum of the interior angles is the same for all pentagons.
- Draw two diagonals from one vertex of your pentagon. How many triangles do these diagonals create?
- How do the interior angles of the triangles relate to the interior angles of the pentagon? Does the same relationship hold for all pentagons?



7. **Reflect** Does the relationship in step 6 confirm your hypothesis about the sum of the interior angles of the pentagon? Explain.
8. Draw another pentagon. Extend one side at each vertex to create an exterior angle. Name and measure the five exterior angles.
9. Find the sum of the exterior angles. Compare this sum to those found by your classmates.
10. **Reflect** Discuss whether the sum of the exterior angles is the same for all pentagons.



Investigate B

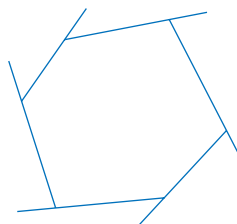
Are there patterns in the angles of polygons?

Method 1: Use Pencil and Paper

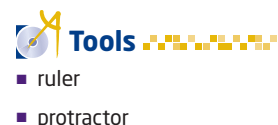
1. Set up a table like the one shown. Enter the results of your previous investigations of angles in polygons. Use the table to record your results during this investigation.

Polygon	Number of Sides	Number of Diagonals From One Vertex	Number of Triangles in the Polygon	Sum of Interior Angles	Sum of Exterior Angles
triangle	3	0	1	180°	360°
quadrilateral					
pentagon					

2. Draw a **hexagon**. Label and measure its interior angles. Find the sum of these angles.
3. Extend one side at each vertex to create an exterior angle. Name and measure these exterior angles. Find the sum of the exterior angles.
4. Draw a **heptagon**. Label and measure its interior angles. Find the sum of these angles.



5. Extend one side at each vertex of your heptagon to create an exterior angle. Name and measure these exterior angles. Find the sum of the exterior angles.
6. **Reflect** Describe any pattern you see for the sum of the exterior angles of a polygon.



hexagon

- a polygon with six sides

heptagon

- a polygon with seven sides

Go to www.mcgrawhill.ca/links/principles9 and follow the links to learn more about the names for polygons.

7. How many diagonals can you draw from a vertex in each polygon? How does the number of diagonals relate to the number of sides? Use a scatter plot or first differences to help identify the relationship.
8. How does the number of triangles formed by the diagonals relate to the number of sides?
9. **Reflect** Explain the pattern in the sums of the interior angles of polygons. Write an equation for this pattern. Describe how you can use this equation to find the sum of the interior angles of any polygon.



computer with *The Geometer's Sketchpad®*

Method 2: Use *The Geometer's Sketchpad®*

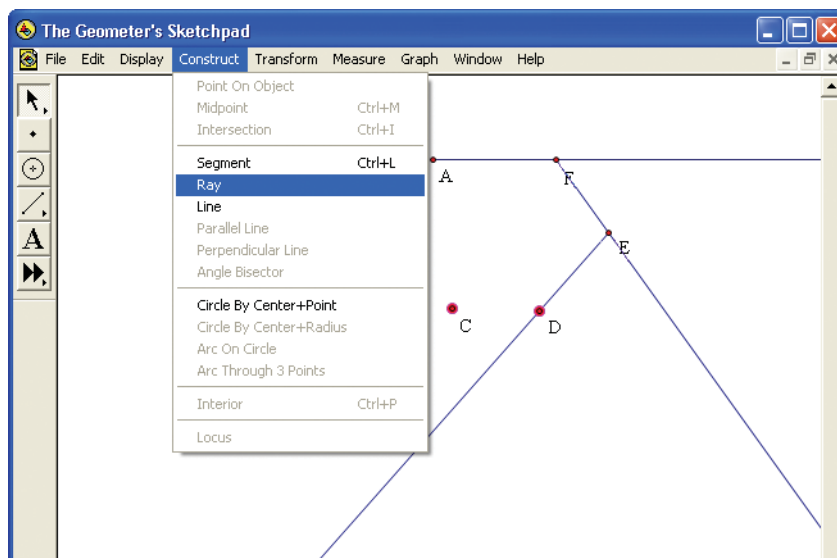
1. Set up a table like the one shown. Enter the results of your previous investigations of angles in polygons. Use the table to record your results during this investigation.

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triangle	3	180°	360°
quadrilateral			
pentagon			

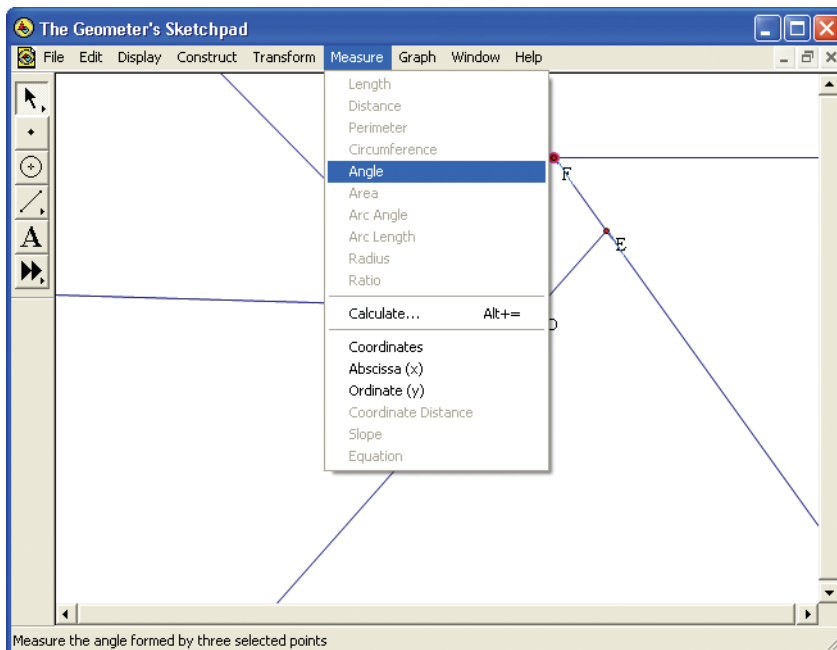
2. Turn on automatic labelling of points. From the **Edit** menu, choose **Preferences**. Click on the **Text** tab, check **For All New Points**, and click on **OK**.
3. Construct a hexagon. Use the **Point Tool** to create six points on the screen. Select point A and then point B. From the **Construct** menu, choose **Ray**. Use the same method to construct rays from B to C, from C to D, from D to E, from E to F, and from F to A.

Technology Tip

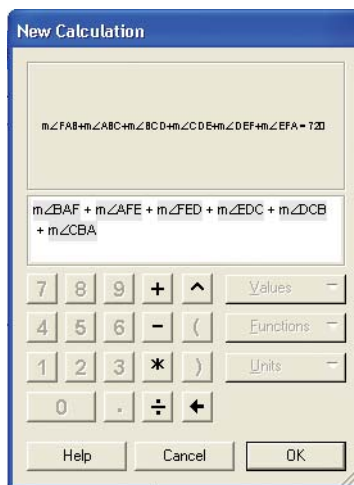
You can also do this investigation using Cabri Jr. on a graphing calculator. For step-by-step instructions, follow the links at www.mcgrawhill.ca/links/principles9.



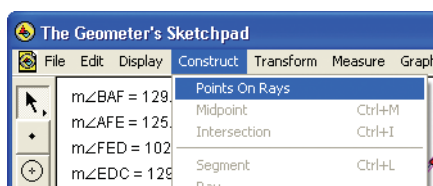
4. To measure $\angle FAB$, select points F, A, and B, in that order. From the **Measure** menu, choose **Angle**. Use the same method to measure the other five interior angles of the hexagon.



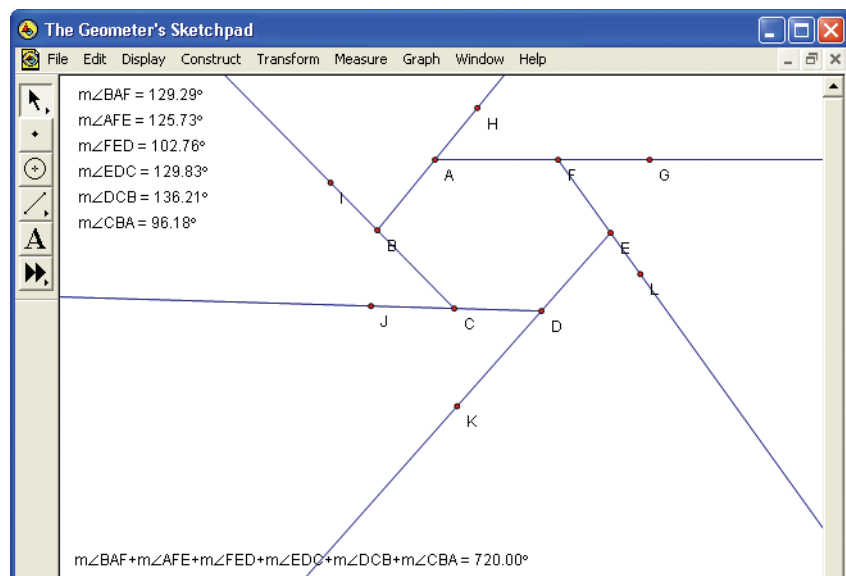
5. From the **Measure** menu, choose **Calculate**. Click on the measure for $\angle FAB$. Click $+$ on the calculator; then, click on the measure for $\angle ABC$. Add the other four interior angles to the calculation. Check whether moving any of the vertices affects the sum of the interior angles.



6. **Reflect** What can you conclude about the sum of the interior angles of any hexagon? Explain your reasoning.
7. Select the six rays. From the **Construct** menu, choose **Points on Rays**. If necessary, drag each point to a location outside the hexagon.



8. Use the Measure and Calculate tools to find the sum of the exterior angles. Check whether moving any of the vertices affects this sum.



9. **Reflect** What can you conclude about the sum of the exterior angles of any hexagon? Explain your reasoning.
10. Use points and rays to construct a heptagon with exterior angles.
11. Use the Measure and Calculate tools to find the sum of the interior angles of your heptagon. Check whether moving any of the vertices affects this sum.
12. **Reflect** What can you conclude about the sum of the interior angles of any heptagon?
13. Select the seven rays. From the **Construct** menu, choose **Points on Rays**. If necessary, drag each point to a location outside the heptagon.
14. Use the Measure and Calculate tools to find the sum of the exterior angles of your heptagon. Check whether moving any of the vertices affects this sum.
15. **Reflect** What can you conclude about the sum of the exterior angles of any heptagon?
16. **Reflect** Describe any pattern you see for the sums of the exterior angles of polygons.
17. **Reflect** Explain the pattern in the sums of the interior angles of polygons. Write an equation for this pattern. Describe how you can use this equation to find the sum of the interior angles of any polygon.



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Example 1 Interior Angles of an Octagon

Calculate the sum of the interior angles of an **octagon**.

octagon

■ a polygon with eight sides

Solution

The equation for the linear relation between the sum of the interior angles of a polygon and its number of sides, n , can be written as

$$\text{Sum of Interior Angles} = 180(n - 2)$$

For an octagon, $n = 8$. So,

$$\begin{aligned}\text{Sum of Interior Angles} &= 180(n - 2) \\ &= 180(8 - 2) \\ &= 180 \times 6 \\ &= 1080\end{aligned}$$

The sum of the interior angles of an octagon is 1080° .

Example 2 Interior Angles in a Regular Octagon

Find the measure of each interior angle of a **regular** octagon.

regular polygon

■ a polygon with all sides equal and all interior angles equal

Solution

Method 1: Use the Formula

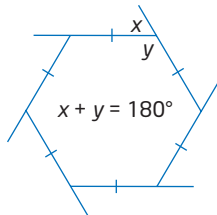
As shown in Example 1, the formula for the sum of the interior angles of a polygon tells you that the sum of the interior angles in an octagon is 1080° . A regular octagon has eight equal interior angles, so the measure of each angle is $\frac{1080^\circ}{8}$ or 135° .

Method 2: Use Supplementary Angles

The sum of the exterior angles of any octagon is 360° . For a regular octagon, these eight angles are equal to each other.

Therefore, the measure of each exterior angle is $\frac{360^\circ}{8}$ or 45° .

At each vertex, the exterior and interior angles sum to 180° . So, the measure of each interior angle is $180^\circ - 45^\circ$ or 135° .



Literacy Connections

A 9-sided polygon is a nonagon. This name comes from *nonus*, the Latin word for ninth.

Example 3 Find the Number of Sides

How many sides does a polygon have if each of its interior angles measures 140° ?

Solution

An n -sided polygon has n interior angles. If each interior angle measures 140° , their sum, in degrees, is $140n$. The sum of the interior angles, in degrees, of any polygon is $180(n - 2)$. Therefore,

$$180(n - 2) = 140n$$

$$180n - 360 = 140n \quad \text{Expand the left side.}$$

$$40n - 360 = 0 \quad \text{Subtract } 140n \text{ from both sides.}$$

$$40n = 360 \quad \text{Add 360 to both sides.}$$

$$n = 9 \quad \text{Divide both sides by 40.}$$

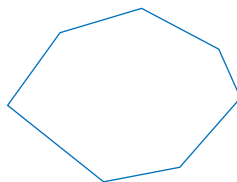
A polygon with interior angles of 140° has 9 sides.

Key Concepts

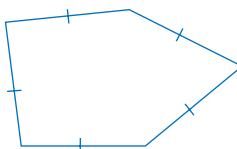
- The sum of the exterior angles of a convex polygon is 360° .
- For a polygon with n sides, the sum of the interior angles, in degrees, is $180(n - 2)$.

Communicate Your Understanding

- C1** Describe how you can determine the sum of the interior angles of this polygon.



- C2** This pentagon has five equal sides. Is it a regular polygon? Why or why not?



Practise

For help with question 1, see Example 1.

- Find the sum of the interior angles of a polygon with
 - 10 sides
 - 15 sides
 - 20 sides

For help with question 2, see Example 2.

- Find the measure of each interior angle of a regular polygon with
 - 7 sides
 - 12 sides

For help with question 3, see Example 3.

- How many sides does a polygon have if the sum of its interior angles is
 - 540° ?
 - 1800° ?
 - 3060° ?
- Copy this table and fill in the missing entries.

Polygon	Number of Sides	Number of Diagonals From One Vertex	Number of Triangles in the Polygon	Sum of Interior Angles
quadrilateral	4	1	2	360°
pentagon		2		
decagon	10			
icosagon	20			

- What properties does a regular polygon have?

Connect and Apply

- Use the formula for the sum of the interior angles of a polygon to show that each interior angle of a square measures 90° .
- A furniture-maker is designing a hexagonal table.
 - At what angle will the adjacent sides of the table meet if its shape is a regular hexagon?
 - Do you think the angles between the adjacent sides of the table will all be equal if one pair of opposite sides are twice as long as the other sides?
 - Check your answer to part b) by making a drawing and measuring the angles.
- Draw a nine-sided polygon.
 - Calculate how many diagonals you can draw from any one vertex of this polygon. Check your answer by drawing all possible diagonals from one of the vertices.
 - Calculate the sum of the interior angles of the polygon. Check your answer by measuring the angles on your drawing.

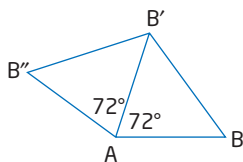


9.
 - a) Find the measure of each interior angle of a regular 10-sided polygon. Use a second method to check your answer.
 - b) Find the measure of each interior angle of a regular 16-sided polygon.
 - c) Find the measure of each interior angle of a regular 20-sided polygon.
 - d) Write an expression for the measure of each interior angle of a regular polygon with n sides.
10.
 - a) How many sides does a Canadian dollar coin have?
 - b) What is the measure of the angle between adjacent sides of the coin?
 - c) Suggest reasons why the Royal Canadian Mint chose this shape.
11. Can you determine the number of sides a polygon has from the sum of its exterior angles? Explain your reasoning.
12. Floor tiles are often in the shape of regular polygons. In order to make a pattern without any gaps, such tiles must have interior angles that divide evenly into 360° . Which three regular polygons have this property?

Literacy Connections

A gazebo is a small structure with open or screened sides that give a wide view.

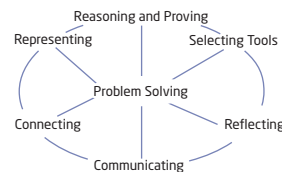
13. This photograph shows the roof of a gazebo. The sides of this gazebo form a regular polygon.
 - a) How many sides does the gazebo have?
 - b) Calculate the angle between adjacent sides of the gazebo.
 - c) Calculate the angle between adjacent roof supports in the photograph.
 - d) Draw a plan of the gazebo.
 - e) Calculate the angle between adjacent supports in the roof of a gazebo with six sides.



14. You can draw regular polygons by constructing angles at the centre point of the figure.
 - a) Construct a line segment AB with a length of 4 cm. Rotate the segment 72° around point A four times. What shape is formed by joining the endpoints of the rotated segments?
 - b) Describe how to use this method to construct a regular octagon.
 - c) What angle would you use for a regular 20-sided figure?
 - d) Describe how the rotation angle for constructing a regular polygon changes as the number of sides increases.

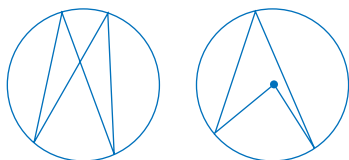
Achievement Check

15. Draw and label an example of each shape, or explain why it is not possible.
- a triangle with one acute exterior angle
 - a triangle with two right angles
 - a quadrilateral with four equal angles
 - a quadrilateral with three obtuse angles
 - a pentagon with two obtuse angles and three acute angles
 - a convex hexagon with five acute angles



Extend

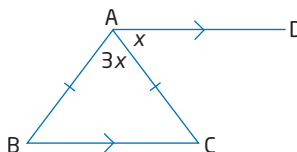
16. Are all regular polygons convex? Justify your answer.
17. Does the formula for the sum of the interior angles apply for concave polygons? Explain your reasoning.
18. Investigate angle relationships in these two diagrams.



19. Use the Internet or a library to find an image of a famous building. Describe how polygons were used in the construction of the building.

20. **Math Contest** The measure of $\angle BCA$ is

- 30°
- 36°
- 45°
- 60°



21. **Math Contest** How many diagonals can you draw in a convex 12-sided polygon?

- 54
- 60
- 108
- 120



Go to
www.mcgrawhill.ca/links/principles9 and follow the links to learn more about famous buildings.